Logarithmic corrections for the percolative properties of the four-state Potts model

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Abstract. Clusters in the Potts model are connected sets of nearest neighbour sites for which the spin variables are in the same state. Droplets can be obtained by adding bonds with probability $p = 1 - \exp(-K)$ between the sites in a cluster (where K is the Potts model's inverse temperature). When $q \to 4$, renormalization group (RG) fixed points describing clusters and droplets will coalesce, leading to logarithmic corrections. We calculate the precise form of these corrections used in a differential RG method. Our predictions are then tested using extensive Monte Carlo calculations. Theory and the simulations are found to be in excellent agreement.

1. Introduction

The percolation transition [1] can be described in terms of geometrical properties of the infinite cluster of occupied sites appearing at the percolation threshold: critical exponents correspond to fractal dimensions [2] of this cluster. This fact raised the question whether one can in general relate critical exponents to fractal dimensions or some suitably defined clusters. As a first guess, one can simply define a cluster in an Ising or q-state Potts model as a connected set of nearest neighbour sites for which the spin variables are in the same state [3]. However, it was soon found [4] that these clusters are not described by thermodynamic exponents: they are too compact. It is now clear that the critical exponents of the above models can be described in terms of the fractal properties of 'droplets' which are obtained by random bond dilution of the clusters. The most important fractal dimensions of two dimensional critical Potts droplets were recently determined using Coulomb gas techniques [8, 9]. These droplets also are relevant to the design of algorithms for Monte Carlo simulation that are efficient near critical points [10].

This still leaves the question open of what are the fractal properties of the naïvely defined clusters. Recently, this problem was solved using renormalization group and conformal invariance arguments, first for the two-dimensional Ising model [11, 12], then for the q-state Potts model [13]. One of the interesting aspects emerging from that work is that for $q = 4$ droplets and clusters coincide. So, for that model geometrical (percolation) and thermodynamic exponents are the same. Independently of this analytical work, extensive Monte Carlo simulations were performed in order to determine the fractal dimension of Potts cluster [14]. While the results of these simulations are in good agreement with the analytic predictions for the Ising and three-state Potts case, the agreement was much less good for the case $q = 4$. In a renormalization group formulation the criticality of droplets and clusters will be described by fixed points which will come together as $q \to 4$, leading to

a marginal eigenvalue and logarithmic corrections [15] at $q = 4$. In this paper we calculate the precise form of these logarithmic corrections and compare them with improved Monte-Carlo data. We find that this explains the disagreement between the analytical and Monte Carlo results.

This paper is organized as follows. In section 2 we give a summary of known results on droplets and clusters. Then, in section 3 we calculate the logarithmic corrections near $q = 4$, using a differential renormalization group procedure. In section 4 we present the results of Monte Carlo simulations for droplets and clusters and compare them with the predictions from section 3. Finally, in section 5 we present our conclusions.

2. Potts clusters, droplets and correlated site-bond percolation

The q-state Potts model [16] is a generalization of the Ising model. At each site i of a lattice there is a variable σ_i which can be in any of q states, $\sigma_i = 1, \ldots, q$. The interaction is given by the reduced Hamiltonian:

$$-\beta H_P = K \sum_{\langle i,j \rangle} (\delta_{\sigma_j \sigma_i} - 1) + h \sum_i (\delta_{\sigma_i, 1} - 1). \tag{2.1}$$

In $d = 2$ and for $h = 0$, this model has a phase transition at some $K_c(q)$ which for $q \leqslant q_c = 4$ is continuous [17].

For each configuration of the $\{\sigma_i\}$ we define clusters as connected sets of nearest neighbour sites in the same state, e.g. in the state 1. The Hamiltonian (2.1) defines a problem of correlated site percolation for these clusters. Indeed, for $K \to \infty$, $H = 0^+$, one expects that all sites will have $\sigma = 1$. Upon increasing the temperature the probability P that a given site of the lattice belongs to the 'infinite' (i.e. lattice spanning) cluster of spins in state 1 will decrease. It was shown rigorously by Coniglio and Peruggi [7] that the percolation threshold for this correlated site percolation problem precisely coincides with $K_c(q)$. For $K \to K_c(q)$ from above, we therefore expect:

$$P(K) \cong A(K - K_c(q))^{\beta_{\text{per}}}. \tag{2.2}$$

The critical exponent β_{per} in general is not the same as the thermodynamic exponent β of the Potts model.

We extend the above percolation problem to a correlated site-bond percolation problem by adding bonds with probability p (uncorrelated) between nearest neighbour sites of the clusters. Sets of nearest neighbour sites for which the spins are in the same state (1), and which are connected with bonds will be referred to as bond-diluted clusters. In order to study these we need a percolative generating function. This generating function f can be defined in terms of the average number (per site) of bond-diluted clusters of s sites, $n_s(K, h, p)$;

$$f(K, h, p, H) = \sum_s n_s(K, h, p) \exp(-sH) \tag{2.3}$$

where H is a 'ghost' magnetic field. An important result for f was obtained by Coniglio and Peruggi [7], who introduced a 'Potts diluted' Potts model (PdP) in which one couples the original Potts variables σ to other n-state Potts variables μ through the Hamiltonian:

$$-\beta H_{\text{PdP}} = -\beta H_P + J \sum_{\langle i,j \rangle} (\delta_{\mu_i \mu_j} - 1) \delta_{\sigma_i, 1} \delta_{\sigma_j, 1} + H \sum_i (\delta_{\mu_i, 1} - 1) \delta_{\sigma_i, 1}. \tag{2.4}$$

It was shown by these authors that $f = dF/dn|_{n=1}$ if one takes

$$p = 1 - \exp(-J) \tag{2.5}$$

and F is the free energy (per site) of the Hamiltonian (4). In the special case $h = H = 0$, $J = K$, it can finally be shown that for $n \to 1$, $H_{PdP} = H_P$ [7], implying that the bond-diluted clusters for these parameter values exhibit the same singularities as are found in the thermodynamic properties of the Potts model of the same q. Following again the authors of [7] we will call the bond-diluted clusters in this case droplets. These are also the objects used in the Swendsen–Wang Monte Carlo algorithm [10]. In the rest of this paper we will be interested in what happens to clusters and droplets near the Potts critical temperature $K = K_c(q)$, $h = 0$. In a recent paper [13], one of the present authors (CV) discussed the behaviour of clusters and droplets under the renormalization group (RG). Using arguments from conformal invariance, general RG arguments, and extending results obtained earlier for the $q = 2$ (Ising) case [11, 12], it was shown that the line $K = K_c(q)$, $h = H = 0$ (hereafter referred to as the critical line) is invariant under renormalization and that along the line there are three fixed points, shown in figure 1. These are the pure Potts-fixed point at $J = p = 0$, the droplet fixed point at $p_c = 1 - \exp(-K_c)$, and the cluster fixed point at an unknown value of p (but $> p_c$). This latter point attracts the critical clusters at $p = 1$. Most interestingly, it was found that for $q \to 4$ the cluster and droplet fixed point coalesce, so that the two kinds of geometrical objects have the same singularities for that q-value. However, it is also known that the coalescence of two fixed points leads to the appearance of logarithmic corrections [15]. These log-corrections form the main subject of this paper. We remark that a coalescence of two fixed points was also observed several years ago in the RG calculation of the thermodynamic properties of the Potts model for $q \to 4$ [18]. We now turn to a discussion of the critical exponents at the cluster and droplet fixed points. These are conveniently parameterized in terms of the variable m which can be related to the number of Potts states q as:

$$\sqrt{q} = 2\cos\left(\frac{\pi}{m+1}\right). \tag{2.6}$$

In the (K, h, J, H)-parameter space the cluster fixed point has three relevant eigenvalues while the droplet fixed point has four. We will be most interested in the eigenvalues corresponding to the variables† H and $u = J - J_*$ where J_* denotes the value of J at the droplet or cluster fixed point. Below, we refer to quantities at the cluster or droplet fixed points by the superscript c or d, respectively; equations where these superscripts are missing hold for both fixed points.

These eigenvalues at the droplet fixed point were determined by Conglio [9], and at the cluster fixed point by one of us [13]. The results are:

$$
y_H^d = \frac{15m^2 + 14m + 3}{8m^2 + 8m} \qquad y_H^c = \frac{15m^2 + 16m + 4}{8m^2 + 8m}
$$
$$
y_u^d = \frac{4m + 1}{2m(m + 1)} \qquad y_u^c = \frac{-4m + 3}{2m(m + 1)}. \tag{2.7}
$$

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† We will use the same notation for the (non)linear scaling fields and for the associated parameters from the Hamiltonian (2.4).
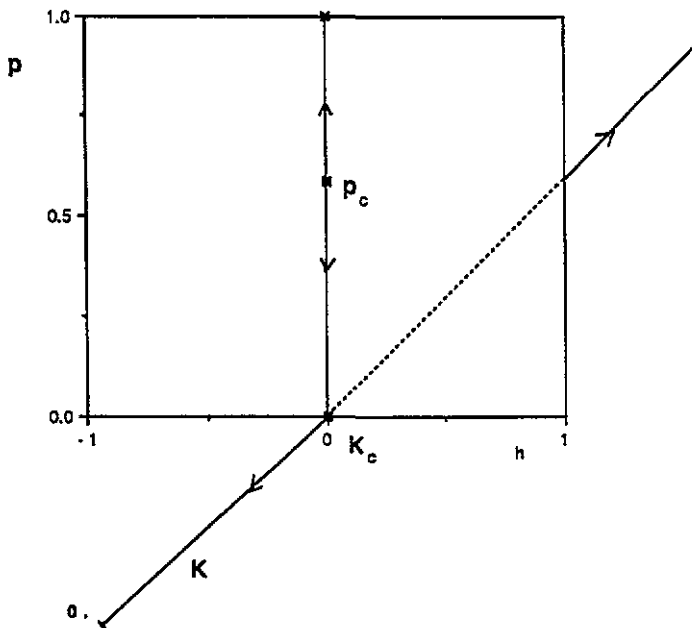
Figure 1. Renormalization group flow for the model (2.4) along the critical line.

The eigenvalues associated with the variables h and $t = K - K_c(q)$ are the same at both fixed points and their values are

$$y_t = \frac{3(m-1)}{2m} \qquad y_h = \frac{15m^2 + 14m + 3}{8m^2 + 8m}. \qquad (2.8)$$

We can now write down a scaling relation for f in the neighbourhood of one of the fixed points. As we will later verify our analytical results with Monte Carlo simulations it is useful to include the finite system size L. We thus have, near both fixed points, the scaling relation:

$$f(t, h, u, H, L) \cong L^{-2} f(L^{y_t} t, L^{y_h} h, L^{y_u} u, L^{y_H} H, 1). \qquad (2.9)$$

A quantity which can easily be measured in a Monte Carlo calculation is the percolative susceptibility χ_p which is defined as:

$$\chi_p(K, h, p, H) = \frac{\partial^2}{\partial H^2} f(K, h, p, H) \qquad (2.10)$$

which from (2.3) is also given by

$$\chi_p(K, h, p, H) = \sum_s s^2 n_s(K, h, p) \exp(-sH).$$

In the following we will be mostly interested in this susceptibility for $K = K_c$, $h = H = 0$ and in a finite system of size L, a quantity for which we will use the notation $\chi_p(p, L)$.

At the droplet (or cluster) fixed point we get from (2.9) that in a finite system χ_p grows as a power law of L:

$$\chi_p(p_c, L) \sim L^{-2+2y_H^d} \qquad (2.11)$$

at the droplet fixed point. For clusters, y_u is irrelevant (2.7), giving:

$$\chi_p(p = 1, L) \sim L^{-2+2y_H^c}. \tag{2.12}$$

At this point it is appropriate to point out that y_H and y_u can be interpreted as fractal dimensions of the infinite percolating cluster (or droplet) which exists at the critical line [9k, 13]. Indeed, y_H is the fractal dimension of the set of points in the cluster (or droplet), while y_u is the fractal dimension of the 'red bonds' [19].

In a recent paper, one of us (JM) [14] performed extensive Monte Carlo simulations of the quantity $\chi_p(p = 1, L)$ for systems with L up to 512. From these data the fractal dimension y_H^c could be extracted. The results were $y_H^c = 1.947 \pm .005$, $1.927 \pm .007$ and $1.918 \pm .015$ for $q = 2$, 3 and 4 respectively. These results have to be compared with the predictions from (2.7) which give $y_H^c = 187/96$, $153/80$ and $15/8$ respectively. While there is excellent agreement for $q = 2$, and good agreement for $q = 3$, there are significant differences for $q = 4$. We believe that this discrepancy is due to logarithmic corrections. It is the purpose of the present paper to calculate the logarithmic corrections to the equation (2.12) at $q = 4$. This is done in section 3. These predictions are then compared with improved Monte Carlo data in section 4.

3. The renormalization group and logarithmic corrections

For $q \to 4$, as discussed in a previous section, the droplet and cluster fixed points will coalesce, leading to an exponent $y_u^{c/d} \to 0$ which in turn leads to logarithmic corrections. In this section we will calculate the precise form of these corrections using techniques that were developed for this purpose in the study of the thermodynamic properties of the Potts model near $q = 4$ [20–22].

Consider a fixed point with the (nonlinear) scaling fields ϕ_i and associated scaling exponents y_i. In particular let ϕ_1 be a scaling field with the marginal eigenvalue $y_1 = 0$. Under an infinitesimal change in length dx these fields will transform according to the RG equations:

$$\frac{d\phi_i}{dx} = y_i \phi_i + \frac{1}{2} \sum_{j,k} a_{ijk} \phi_j \phi_k \tag{3.1}$$

where $a_{ijk} \neq 0$ only when $y_i = y_j + y_k$. The coefficients a_{ijk} are proportional to the operator product expansion coefficients [23]. These can in principle be determined from the conformal field theory describing the critical system [24, 25]. Unfortunately, we do not know enough about the theory describing the critical Potts clusters to determine all the coefficients a_{ijk}. Nevertheless, we can determine the form of the RG equations as follows.

For the four-state Potts model, we have from (2.8) ($m \to \infty$)

$$y_H^d = y_h = y_H^c = 15/8 \qquad y_t = 3/2 \qquad y_u^d = y_u^c = 0.$$

Then solely on the basis of the requirement $y_i = y_j + y_k$ we can write (3.1) as:

$$\frac{d\phi_i}{dx} = y_i \phi_i + \frac{1}{2} a_{ii1} \phi_i \phi_1 \tag{3.2}$$

where $\phi_1 = u$, i.e. the scaling field with $y_1 = y_u = 0$. Further, we expect physically that the scaling of the thermodynamic fields t and h cannot be influenced by geometric properties

and vice versa. The fields t and h thus still obey the RG equations given by, e.g. Cardy *et al* [21].

Thus we only have to write down the RG equations for the remaining fields u and H, in the neighbourhood of $q = 4$, near the multicritical point at which cluster and droplet critical lines come together to first order in $\epsilon = q - 4$. Following the procedure described in [21] we obtain:

$$\frac{du}{dx} = \alpha(u^2 + \epsilon) \tag{3.3a}$$

and

$$\frac{dH}{dx} = (y_H(q = 4) + \beta u)H \tag{3.3b}$$

where α and β are constants which remain to be determined (recall (2.7) which gives $y_H(q = 4) = 15/8$). Notice that for $\epsilon = 0$, (3.3) is of the form (3.2). In a finite system, the size b will be normalized according to:

$$\frac{db}{dx} = -b. \tag{3.3c}$$

The equations (3.3) have two fixed points: they are at $H = b^{-1} = 0$ and $u = \pm\sqrt{-\epsilon}$. These, we identify as the droplet (+) and cluster (−) fixed point. Linearization of (3.3) at these fixed points gives the following expansions for the critical exponents y_u and y_H:

$$y_u(q) = \pm 2\alpha\sqrt{-\epsilon} + \dots \qquad \text{and} \qquad y_H(q) = y_H(q = 4) \pm \beta\sqrt{-\epsilon} + \dots. \tag{3.4}$$

If we now expand the equations (2.7) to first order in $\sqrt{-\epsilon}$ and compare with (3.4) we find $\alpha = 1/2\pi$ and $\beta = -1/16\pi$. Having thus fixed the constants α and β we can integrate the equations (3.3). For a system of size L and with initial values u_0, H_0 for the scaling fields u and H we have:

$$u(L) = u_0 \left(1 - \frac{u_0}{2\pi} \ln L\right)^{-1} \qquad \text{and} \qquad H(L) = H_0 L^{15/8} \left(1 - \frac{u_0}{2\pi} \ln L\right)^{1/8}. \tag{3.5}$$

Under this rescaling, the free energy f will transform as:

$$f(0, 0, u_0, H_0, L) \cong L^{-2} f(0, 0, u(L), H(L), 1). \tag{3.6}$$

We have placed ourselves already at the 'thermodynamic' fixed point $t = h = 0$. Equation (3.6) replaces (2.9) at $q = 4$. We then find that the susceptibility (2.10) for clusters at $p = 1$ (i.e. with $u_0 < 0$) diverges with L as:

$$\chi_P(q = 1, L) \sim L^{7/4} \left(1 - \frac{u_0}{2\pi} \ln L\right)^{1/4}. \tag{3.7}$$

We can thus see that if we use equation (2.11) to analyse a set of numerical data, instead of the correct result (3.7) we will get an estimate for the fractal dimension of clusters y_H^c which is too big. This explains qualitatively the discrepancy between the prediction (2.7) and the numerical results of [14], for $q = 4$.

In the next section we will compare (3.7) with improved Monte Carlo data. Here we point out some further consequences of our RG equations. First, we can make $u_0 = 0$ in equation (3.6) by putting ourselves at the droplet (= cluster) fixed point. For $q = 4$, and on the square lattice, this fixed point is located at (one has $K_c = \ln \sqrt{q} + 1$ on the square lattice [16])

$$p = p_c = 1 - \exp(-K_c) = 1 - \frac{1}{2+1} = \frac{2}{3}. \tag{3.8}$$

Of course, for this p-value, the equations (3.7) and (2.11) coincide. In the next section we will also verify this prediction with Monte Carlo simulations.

Finally, we derive some results for the percolation probability P as a function of q. At the Potts critical temperature (2.2) indicates that $P \to 0$. This is only true when $q \leqslant 4$. For $q > 4$ the two dimensional Potts model has a first-order transition at $K_c(q)$, and we thus have $P(K_c(q)) > 0$. We now briefly calculate how P goes to zero when $q \to 4$ from above. We can write

$$P(K_c) = \frac{\partial f}{\partial H}(t \to 0^+) - \frac{\partial f}{\partial H}(t \to 0^-). \tag{3.9}$$

To calculate P we thus need to extend (3.6) to the case where $t \neq 0$, for infinite system size.

Under a rescaling with a factor $b = e^x$, the scaling relation for f then becomes:

$$f(t_0, 0, u_0, H_0) \cong e^{-2x} f(t(x), 0, u(x), H(x)) \tag{3.10}$$

from which we have

$$\frac{\partial f}{\partial H_0}(t_0, 0, u_0, H_0) \cong e^{-2x} \frac{\partial f}{\partial H(x)}(t(x), 0, u(x), H(x)) \frac{dH(x)}{dH_0}. \tag{3.11}$$

In calculating (3.9) we use the fact that the value $t = 0^+$ will be renormalized (when $x \to \infty$) to a zero temperature fixed point where $P = \partial f/\partial H = 1$, while $t = 0^-$ will renormalize to a high temperature fixed point with $P = 0$. Using (3.9) and (3.11) we get

$$P(K_c) = \lim_{x \to \infty} e^{-2x} \frac{dH(x)}{dH_0}. \tag{3.12}$$

The derivative in (3.12) can be obtained by combining the equations (3.3) into

$$\frac{dH}{H} = \frac{15/8 - 1/16\pi(u)}{(u^2 + \epsilon)} 2\pi \, du$$

so that after integration we get

$$P(K_c) = \lim_{x \to \infty} \left[\exp - \int_{u_0}^{u(x)} \frac{\pi/4 + 1/8u}{u^2 + \epsilon} \, du \right] \tag{3.13}$$

from which we obtain (using $u_0 < 0$, and $\lim_{x \to \infty} u(x) > 0$) that for $\epsilon \to 0$ from above

$$P(K_c) \sim \exp -\frac{\pi^2}{4\sqrt{\epsilon}} (1 + O(\epsilon)). \tag{3.14}$$

This result can be compared with similar results which exists for the latent heat, or for the magnetization for $q \to 4^+$ [21].

4. Monte Carlo calculation of susceptibilities

In this section, we show that the preceding analytical conclusions are verified by numerical Monte-Carlo (MC) calculations of percolative and magnetic susceptibilities of the critical $q = 4$ Potts model. These calculations employ methods similar to those previously used by one of the authors (JM) although much smaller statistical errors have been achieved in this study by the use of much longer MC integrations. We have used the Swendsen–Wang (SW) algorithm [10] to do the Monte Carlo simulation. A single move is defined as follows: starting from a spin state $\{\sigma_i\}$, bonds are added between neighbouring like spins with probability $1 - e^{-K}$, while no bonds are added between neighbouring unlike spins. The sets of spins that are connected together by bonds thus just coincide with the droplets. Then each droplet is assigned a spin from 1 to q randomly: changing each spin in a droplet to the assigned value for that droplet completes the SW move. It is easy to show that the probability of observing a given spin state $\{\sigma_i\}$ in an infinite sequence of such moves is proportional to the Boltzmann weight $\exp(-\beta H_p\{\sigma_i\})$. In a finite system, the SW clusters must have a characteristic scale of the system size. Thus this algorithm rapidly (in terms of numbers of SW moves) decorrelates distant sites. It turns out that the added computation needed to construct the cluster is more than compensated for by a vast reduction in the correlation time, making the algorithm preferable to a single-site-flip Metropolis MC calculation.

We consider a critical 4-state Potts model with periodic boundary conditions on an $L \times L$-site square lattice. The magnetic susceptibility χ may be expressed as an integral of the connected two-point magnetic correlator of spins σ_i and σ_j, $g(i, j) \equiv \langle \delta_{\sigma_i,\sigma_j} \rangle - q^{-1}$:

$$\chi(L) = L^{-2} \sum_{i,j} g(i, j). \tag{4.1}$$

This is just the mean-squared fluctuation of the number of spins in each spin state

$$\chi(L) = L^{-2} \sum_{s=1,\dots,q} (\langle m_s^2 \rangle - \langle m_s \rangle^2) \tag{4.2}$$

where $m_s = \sum_i \delta_{s,\sigma_i}$. At criticality (ignoring logarithmic corrections for the moment) $g(i, j) \approx |r_i - r_j|^{2y_h-4}$, so we expect $\chi(L) \approx L^{2y_h-2}$, where $y_h = 1.875$ for $q = 4$. The percolative susceptibility, defined in (2.10), can be determined in a Monte Carlo calculation as the average sum of the squares of the 'masses' s of clusters that are obtained by connecting, with probability p neighbouring like spins:

$$\chi_p(p, L) = L^{-2} \left\langle \sum_{\text{clusters } c} s_c^2 \right\rangle \approx L^{2y_H-2} \tag{4.3}$$

where again the scaling behaviour indicated (from (2.11) and (2.12)) ignores logarithmic corrections. It should be noted that the $\chi_p(p, L)$ measurement procedure is independent of the SW integration, i.e. $p \neq 1 - e^{-K}$ in general.

We have carried out computations with these susceptibilities for L between 4 and 240 using the following scheme for each value of L. First, the 'correlation time' $\tau(L)$ was measured: we imagine that this is the typical number of SW moves between two spin configurations that are predominantly 'uncorrelated'. What exactly this means is not entirely clear: however, one can define this time to be the characteristic decay time of the autocorrelation function of some operator. We chose to measure the autocorrelation of
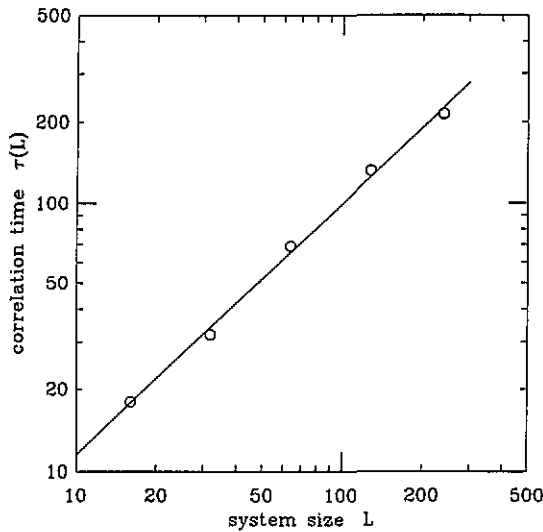
Figure 2. Correlation times τ as a function of system size L for critical $q = 4$ Potts model. The straight line indicates a fit of the form $\tau = AL^z$, with amplitude $A = 1.32$ and dynamic exponent $z = 0.94$.

fluctuations of the sum of the squares of the masses of the $p = 1$ clusters (i.e. the value of $\sum s_c^2$ of (4.3)). This was done by first equilibrating a spin configuration by applying $\approx 20\,000$ SW steps, and then calculating the autocorrelation function for a series of 16 384 consecutive s_k values. A decaying autocorrelation function is obtained: we define τ to be the number of SW steps at which the autocorrelation decays to $1/e$ of its peak value.

Figure 2 shows $\tau(L)$: a (very approximate) dynamic exponent z may be measured by fitting the form $\tau(L) \approx AL^z$ to the data, giving $A = 1.32(5)$ and $z = 0.94(4)$. It would be straightforward to determine τ more accurately, but since we just want to ensure that we are measuring equilibrium, the rough accuracy of this study is sufficient. The main result of this graph is that the correlation time for $L = 240$, the largest system studied, is of order 200.

We measured three susceptibilities: (*a*) magnetic susceptibility $\chi(L)$, (*b*) $p = 1$ percolative susceptibility $\chi_p(p = 1, L)$, and (*c*) $p = 2/3$ percolative susceptibility $\chi_p(p = 2/3, L)$. For each of $L = 4, 8, 12,\ldots,60$, a set of data (one value of each of the χ's) was obtained by (*a*) equilibrating for $T_E = 8192$ SW steps, and (*b*) averaging the susceptibility operators over $T_R = 20\,000$ further SW steps. This procedure was then repeated for each L to obtain $N_R = 110$ statistically independent samples of the susceptibilities, using the end configuration of one run as the starting configuration for the next. The resulting data sets were used to obtain final estimates for the mean and standard deviations of the susceptibility distributions. Our error estimates are the standard deviations divided by $N_R^{1/2}$, appropriate for statistically independent samples. For $L = 64, 80, 96, 112$ and 128, the same procedure was carried out with $T_E = 16\,384$, $T_R = 40\,000$ and $N_R = 54$; for $L = 192$ and $L = 240$ we used $T_E = 32\,768$, $T_R = 40\,000$, and $N_R = 78$. We note that these run times are at least 10 times as large as the correlation times.

The averaged data for the $p = 1$ percolative susceptibility, and for the magnetic susceptibility, are shown in figure 3. The errors are invisible as they are smaller than the plot symbols. The power law $L^{7/4}$ is shown for comparison, and appears as a straight
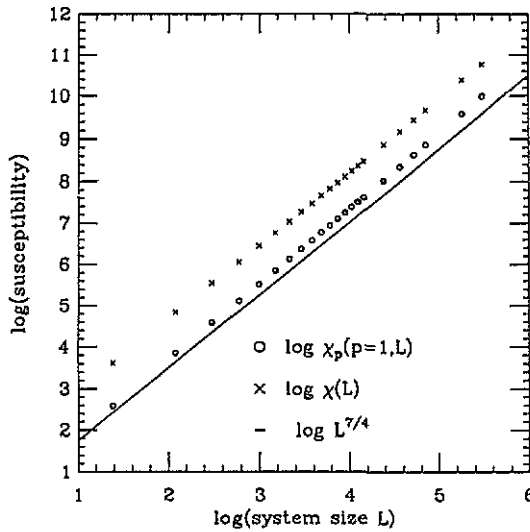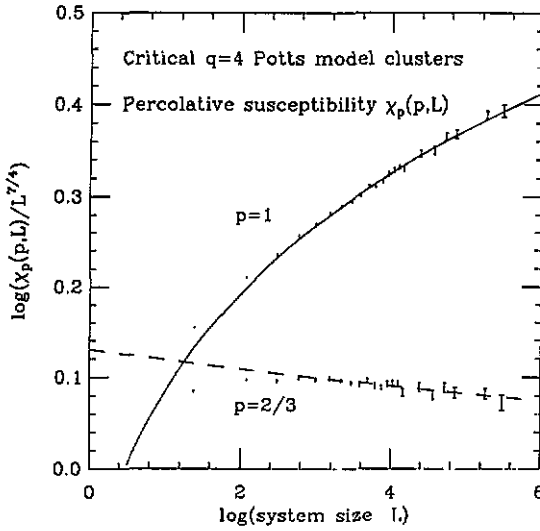
Figure 3. Percolative susceptibility for bond probability $p = 1$ (circles) and magnetic susceptibility (crosses) for critical $q = 4$ Potts model. Error bars are smaller than the symbols. A line of slope 1.75 is plotted beside the data for comparison.

line on the log–log plot. By eye, it appears that $\chi_p(p = 1)$ has a slope in excess of 7/4, while χ appears to have a slope very close to 7/4. Naïvely extracting exponents for the data sets by least-squares linear regression of the susceptibility logarithms against $\ln L$ gives: $\chi_p(p = 1, L) \propto L^{1.807}$ [12] and $\chi(L) \propto L^{1.746}$ [6], giving estimates of $y_H = 1.904(6)$ and $y_h = 1.873(3)$. We see that y_h is in perfect accord with the exact results $y_h = 15/8$ but y_H is significantly larger than the conformal invariance result for the $q = 4$ clusters $y_H = 15/8$.

We note that the apparent value of $y_H = 1.904(6)$ is somewhat smaller than the result ($1.918(15)$) of the previous numerical study [14], although their error ranges do overlap. This difference could be due to the data for $L = 256$ and $L = 512$: fitting a power law to the portion of the old data set with $L < 256$ yields $y_H = 1.909(14)$, in closer agreement with the new data. A case could be made to ignore those data, as they have much larger errors. There is extremely close agreement of the old and new data in the range $L \leqslant 80$. In any case, the approach of fitting a pure power law to χ_p is inappropriate if there are appreciable logarithmic corrections.

The discrepancy between the apparent power-law or χ_p and the exact exponent may be resolved by noting that $\chi_p(p = 1, L)$ has the scaling form (3.7). The first thing to do is to get rid of an overall factor of $L^{7/4}$; in figure 4 we show $\ln(\chi_p(p = 1, L)/L^{7/4})$ and $\ln(\chi_p(p = 2/3, L)/L^{7/4})$ plotted versus $\ln L$. Now the errors are visible (note the change in scale compared with figure 3), and we can see that we can resolve definite structure in $\chi_p/L^{7/4}$. For clusters ($p = 1$) we see that after removal of the $L^{7/4}$, there is a component of χ_p that slowly increases with L, with a decreasing slope–behaviour consistent with (3.7) with a negative value of u_0. A nonlinear-least-squares fit of the data for $L \geqslant 24$ to the form $\chi_p(p = 1, L) = AL^{7/4}(1 - u_0 \ln L/2\pi)^{1/4}$ results in the fit parameters $A = 0.894(2)$ and $u_0 = -7.49(8)$. This curve is shown in figure 4, and passes through the data points within the error bounds for $L \geqslant 24$. The lower data points in figure 4 show the percolative susceptibility for the 'droplets' ($p = 2/3$): in agreement with the prediction of section 3, the correction to the power-law behaviour is greatly reduced: there is now a correction to

Figure 4. Percolative susceptibility for clusters ($p = 1$, upper data) and for droplets ($p = 2/3$, lower data) for critical $q = 4$ Potts model after removal of the leading power law $L^{7/4}$. The smooth curve is a fit of the scaling form $A(1 - u_o \log L/2\pi)^{1/4}$ to the data for $\chi_p(p = 1)/L^{7/4}$ for $L \geqslant 24$: the parameters $A = 0.894(2)$ and $u_0 = -7.49(8)$ are obtained. The lower data show that this correction is greatly suppressed when we set $p = 2/3$. The dashed curve shows the logarithmic correction for the magnetic susceptibility, $B_P(1 - v_{0,P} \ln L/2\pi)^{-1/8}$ for $B_P = 1.139(3)$ and $v_{0,P} = -0.58(6)$, obtained by fitting to the $\chi_p(p = 2/3)/L^{7/4}$ data for $L \geqslant 24$.

scaling that slowly decreases with L.

Further evidence for the prediction that $\chi_p(p = 2/3)$ describes droplets is shown in figure 5, where we plot the logarithm of the magnetic susceptibility with the leading power law divided out, $\ln[\chi(L)/L^{7/4}]$, versus $\ln L$. As can be seen by comparison with figure 4, apart from an overall multiplicative factor (a vertical shift on the log–log plots, the behaviours of $\chi/L^{7/4}$ and $\chi_p(p = 2/3)/L^{7/4}$ are nearly identical. This agrees with the picture presented in section 3, i.e. that for $p = 2/3$ the percolative and magnetic properties coincide. Cardy *et al* [22] have shown that there are logarithmic corrections to the magnetic susceptibility of the critical $q = 4$ Potts model with form $\chi(L)/L^{7/4} = B(1 - v_0 \ln L/2\pi)^{-1/8}$. We find that we may fit our $\chi(L)$ data for $L \geqslant 24$ with $B = 3.41(1)$ and $v_0 = -0.57(6)$: the fit scaling function is shown as a solid line in figure 5. Furthermore, we may fit this scaling form to the $L \geqslant 24$ region of the $\chi_p(p = 2/3)$ data of figure 4, giving $B_p = 1.139(3)$ and $v_{0,P} = -0.58(6)$: the resulting fit is shown in figure 4 as a dashed line. The agreement $v_0 = v_{0,P}$ further supports the arguments of section 3.

5. Conclusions

In this paper, we have presented a derivation of the form of logarithmic corrections for the percolative properties of the critical $q = 4$ Potts model. When the probability p (that a bond is made between neighbours with equal spins) used to compute χ_p is $p = 1$ clusters of fractal dimension 15/8 are revealed. However, for a system of finite size L, logarithmic
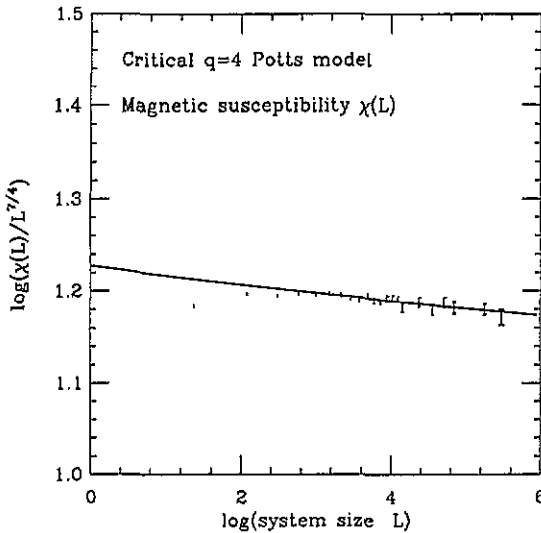
Figure 5. Magnetic susceptibility for critical $q = 4$ Potts model after removal of the leading power law $L^{7/4}$. A weak correction to leading scaling is observed: apart from an overall multiplicative constant (an addictive shift on the log-log plot) we see that this correction is almost identical to that obtained for $\chi_p(p = 2/3)$ in figure 4. The solid curve indicates the magnetic logarithmic correction $B(1 - v_0 \ln L/2\pi)^{-1/8}$ for $B = 3.41(1)$ and $v_0 = -0.57(6)$, obtained by fitting to the $\chi/L^{7/4}$ data for $L \geqslant 24$.

corrections to the leading $L^{7/4}$ power-law scaling of χ_p have the property that they make it *appear* to have a larger exponent, corresponding to a slightly larger fractal dimension of about 1.91. We have carried out large-scale numerical calculations of the finite-size scaling of χ_p, and we find the results to agree with the analytical scaling prediction, resolving this apparent discrepancy.

Furthermore, the analytical theory predicts that at $q = 4$, the clusters and droplets (the objects accounted for by χ_p with $p_c = 1 - e^{-K_c}$) are described by the same scaling exponents (they have the same fractal dimension of 15/8), and also that the droplet χ_p should not have the logarithmic correction described above, by virtue of the fact that the droplet fixed point is located at p_c. This was also verified: corrections to leading scaling are much smaller than for the clusters ($p = 1$). The remaining corrections are identical to those for the *magnetic* susceptibility χ. This last conclusion is remarkable once one considers that these two quantities are averages of qualitatively differently defined operators: χ is simply calculated from the spins, while χ_p must be computed in a highly non-local way, by decomposing the lattice of spins into percolation clusters!

Percolative properties of critical Potts models in $d = 2$ provide a simple example of how one can find (relevant) operators in one model that are not contained in the conformal field theory describing the thermodynamical fixed point. In the $q = 4$ case the percolative and magnetic correlators have equal scaling dimensions, but logarithmic corrections to the percolative corrections (for e.g. $p = 1$) distinguish them from the magnetic correlations: the form of this logarithmic correction to χ_p remains as a reminder that there are scaling properties of the spins of the critical Potts model that are not described by the RG theory of the thermodynamical properties.

Acknowledgments

References

[1] Essam J W 1980 *Rep. Prog. Phys.* **43** 835
[2] Mandelbrot B B 1982 *The Fractal Geometry of Nature* (San Franscisco: Freeman)
[3] Fisher M E 1967 *Physics NY* **3** 255
[4] Binder K 1976 *Ann. Phys., NY* **98** 390
[5] Coniglio A, Nappi C M, Peruggi F and Russo L 1977 *J. Phys. A: Math. Gen.* **12** 205
[6] Coniglio A and Klein W 1980 *J. Phys. A: Math. Gen.* **13** 2775
[7] Coniglio A and Peruggi F 1982 *J. Phys. A: Math. Gen.* **15** 1873
[8] Saleur H, Duplantier B 1987 *Phys. Rev. Lett.* **58** 2325
[9] Coniglio A 1989 *Phys. Rev. Lett.* **62** 3054
[10] Swendsen R H and Wang J S 1987 *Phys. Rev. Lett.* **58** 86
[11] Stella A L and Vanderzande C 1989 *Phys. Rev. Lett.* **62** 1067
[12] Vanderzande C and Stella A L 1989 *J. Phys. A: Math. Gen.* **22** L445
[13] Vanderzande C 1992 *J. Phys. A: Math. Gen.* **25** L75
[14] Marko J F 1992 *Phys. Rev. B* **45**
[15] Wegner F 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) p 8
[16] Wu F Y 1982 *Rev. Mod. Phys.* **54** 2325
[17] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L94
[18] Nienhuis B, Berker A N, Riedel E K and Schick M 1979 *Phys. Rev. Lett.* **43** 737
[19] Coniglio A 1982 *J. Phys. A: Math. Gen.* **15** 3829
[20] Nauenberg M and Scalapino D J 1980 *Phys. Rev. Lett.* **44** 837
[21] Cardy J L, Nauenberg M and Scalapino D J 1980 *Phys. Rev. B* **22** 2560
[22] Stella A L and Vanderzande C 1987 *J. Phys. A: Math. Gen.* **20** 3001
[23] Cardy J L 1986 *J. Phys. A: Math. Gen.* **19** L1093
[24] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *J. Stat. Phys.* **34** 763
[25] Cardy J 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J Lebowitz (New York: Academic) p 55